Rossmoyne SHS Mathematics Department	MATHEMATICS SPECIALIST 3CD	Semester 1 2010 EXAMINATION

NAME:	
TEACHER:	Mr Birrell Mr Whyte Mr Longley

Section Two: Calculator-assumed

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for this section: 100 minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

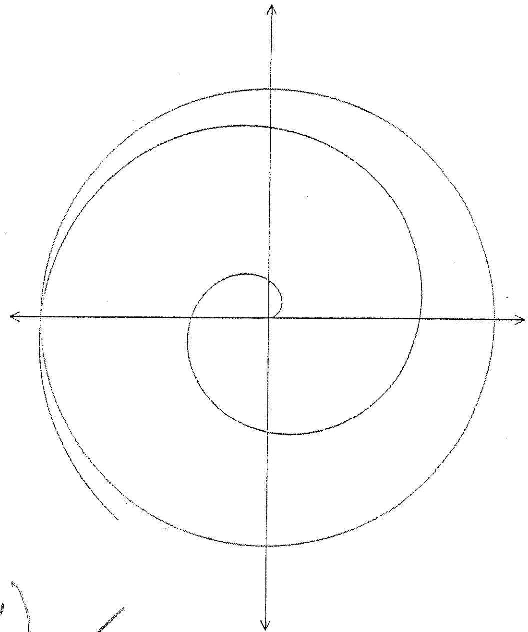
To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

1. [3 marks]
Consider the polar graphs of a spiral and a circle on the right. If the Cartesian equation of the circle is $x^2 + y^2 = \pi^4$ determine the polar equations of the spiral.



Circle : $r = \pi^2$

Spiral : $r = k\theta$ ✓

Intersection at $(\pi^2, 3\pi)$ ✓

$$\pi^2 = k(3\pi)$$

$$\therefore k = \frac{\pi}{3}$$

Eq^n is $r = \frac{\pi\theta}{3}$ ✓

2. [3 marks]
Points A, B and C have position vectors $6\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$, $4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$ and $5\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ respectively. Prove that A, B and C are collinear.

$$\vec{AB} = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

✓✓

$$\therefore \vec{AB} = 2\vec{AC}$$

Since vectors $\vec{AB}$ + $\vec{AC}$ are parallel and both contain point A, the points A, B + C are collinear.

✓

3. [3,2 marks]

A curve is defined by the parametric equations: $x = t^2 \sin 3t$ and $y = t^2 \cos 3t$

a) Find $\frac{dy}{dx}$ in terms of t.

$$\frac{dx}{dt} = 2t\sin(3t) + 3t^2\cos(3t)$$
$$= t(2\sin(3t) + 3t\cos(3t))$$
$$\frac{dy}{dt} = 2t\cos(3t) - 3t^2\sin(3t)$$
$$= t(2\cos(3t) - 3t\sin(3t))$$
$$\frac{dy}{dx} = \frac{2\cos(3t) - 3t\sin(3t)}{2\sin(3t) + 3\cos(3t)}$$

b) Show that if the curve defined by these parametric equations is horizontal at any point, then $\tan 3t = \frac{2}{3t}$.

If horizontal, $\frac{dy}{dx} = 0$.

$$\therefore 0 = 2\cos(3t) - 3t\sin(3t)$$
$$\therefore 3t\sin(3t) = 2\cos(3t)$$
$$\therefore \frac{\sin(3t)}{\cos(3t} = \frac{2}{3t}$$
$$\therefore \tan(3t) = \frac{2}{3t}$$

4. [4 marks]

Given $y\ln x - y^2 = 2$ prove $\frac{dy}{dx} = \frac{-y^2}{x(2-y^2)}$.

$$\frac{d}{dx}\left\{ y\ln x - y^2 = 2 \right\}$$

$$\frac{dy}{dx}\ln x + \frac{y}{x} - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(\ln x - 2y) = -\frac{y}{x}$$

$$\frac{dy}{dx} = \frac{-y}{x(\ln x - 2y)}$$

$$= \frac{-y^2}{x(y\ln x - 2y^2)}$$

$$= \frac{-y^2}{x(2+y^2-2y^2)}$$

$$= \frac{-y^2}{x(2-y^2)}$$

5. [4 marks]

Prove: $\frac{\sin 3\theta}{\cos\theta} = \tan\theta(2\cos 2\theta + 1)$

$$LHS = \frac{1}{\cos\theta}\left(\sin(2\theta+\theta)\right)$$

$$= \frac{1}{\cos\theta}\left(\sin 2\theta \cos\theta + \sin\theta\cos 2\theta\right)$$

$$= \frac{1}{\cos\theta}\left(2\sin\theta\cos^2\theta + \sin\theta\cos 2\theta\right)$$

$$= \frac{\sin\theta}{\cos\theta}\left(2\left(\frac{\cos 2\theta + 1}{2}\right) + \cos 2\theta\right)$$

$$= \tan\theta(\cos 2\theta + 1 + \cos 2\theta)$$

$$= \tan\theta(2\cos 2\theta + 1)$$

$$= RHS$$

6. [3 marks]
Find the acute angle that the vector $\underset{\sim}{a} = 3\underset{\sim}{i} + 5\underset{\sim}{j} - 7\underset{\sim}{k}$ makes with the z-axis.

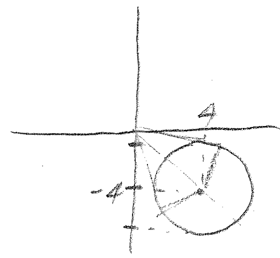
$\underset{\sim}{a} \cdot \underset{\sim}{k} = -7$ ✓

$\therefore -7 = (1)\sqrt{83}\cos\theta$ ✓

$\theta = \cos^{-1}\left(\frac{-7}{\sqrt{83}}\right)$

$= 2.447$ (3dp) (140.2°) ✓

7. [1,1,2,2 marks]
For $\{z : |z - 4 + 4\mathrm{i}| = 3\}$ determine



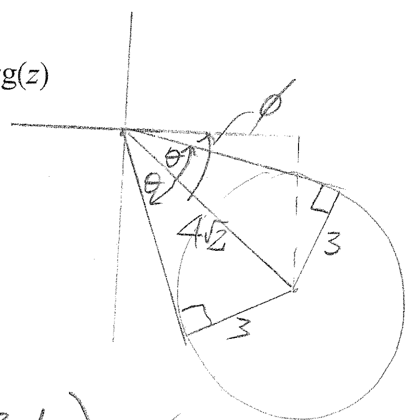
a) The minimum possible value of Im(z)

Min $\operatorname{Im}(z) = -7$ ✓

b) The maximum possible value of $|z|$

Max $|z| = 4\sqrt{2} + 3$ ✓

c) The minimum possible value of arg(z)



$\phi = \frac{\pi}{4}$

$\sin\theta = \frac{3}{4\sqrt{2}}$ ✓

Min $\arg(z) = -\left(\frac{\pi}{4} + \theta\right)$

$= -1.344$ (3dp) ✓

d) The cartesian equation of the curve.

$(x-4)^2 + (y+4)^2 = 9$ ✓✓ (-1 per error)

8. [4 marks]
Differentiate $\sin(3x)$ from first principals

$$\frac{d}{dx}\{\sin(3x)\} = \lim_{h\to 0}\left\{\frac{\sin 3(x+h) - \sin 3x}{h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{\sin 3x\cos 3h + \sin 3h\cos 3x - \sin 3x}{h}\right\}$$

$$= \lim_{h\to 0}\frac{\sin 3x\cos 3h - \sin 3x}{h} + \lim_{h\to 0}\frac{\sin 3h\cos 3x}{h}$$

$$= \sin 3x\left(\lim_{h\to 0}\frac{\cos 3h - 1}{h}\right) + \cos 3x\lim_{h\to 0}\frac{\sin 3h}{h}$$

$$= \sin 3x\left(3\lim_{h\to 0}\frac{\cos 3h - 1}{3h}\right) + \cos 3x\left(3\lim_{h\to 0}\frac{\sin 3h}{3h}\right)$$

$$= \sin 3x \times 3 \times 0 + \cos 3x \times 3 \times 1$$

$$= 3\cos 3x$$

9. [5 marks]

Prove, by contradiction, that $\frac{\log 7}{\log 2}$ is irrational.

If rational, $\frac{\log 7}{\log 2} = \frac{a}{b}$ where a & b are integers.

$\therefore \log_2 7 = \frac{a}{b}$

$\therefore 7 = 2^{\frac{a}{b}}$

$\therefore 7^b = 2^a$

Since all powers of 7 are odd
This cannot be true, hence we have a contradiction.

$\therefore \frac{\log 7}{\log 2}$ is irrational.

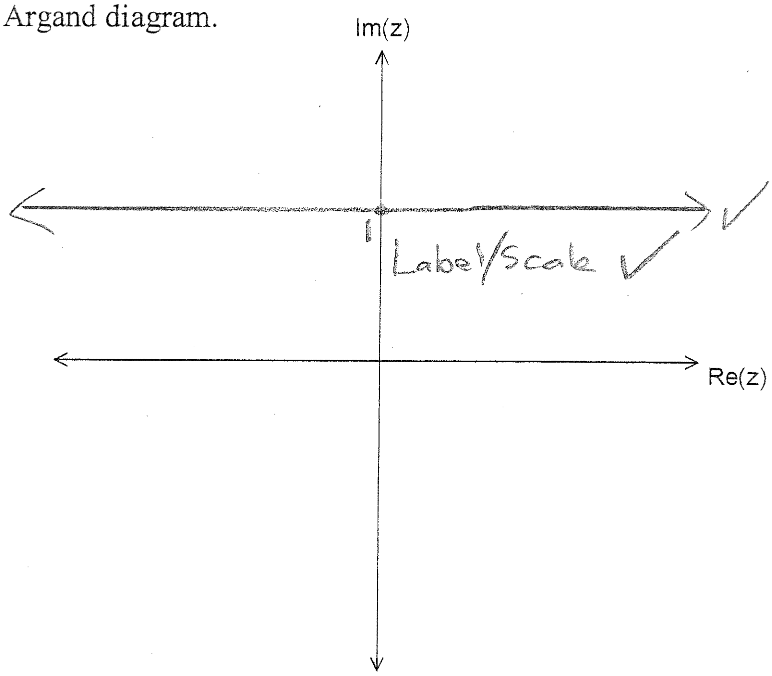
10. [2,3 marks]
Represent the following on an Argand diagram.

a) $\{z : z - \bar{z} = 2i\}$

let $z = x + iy$

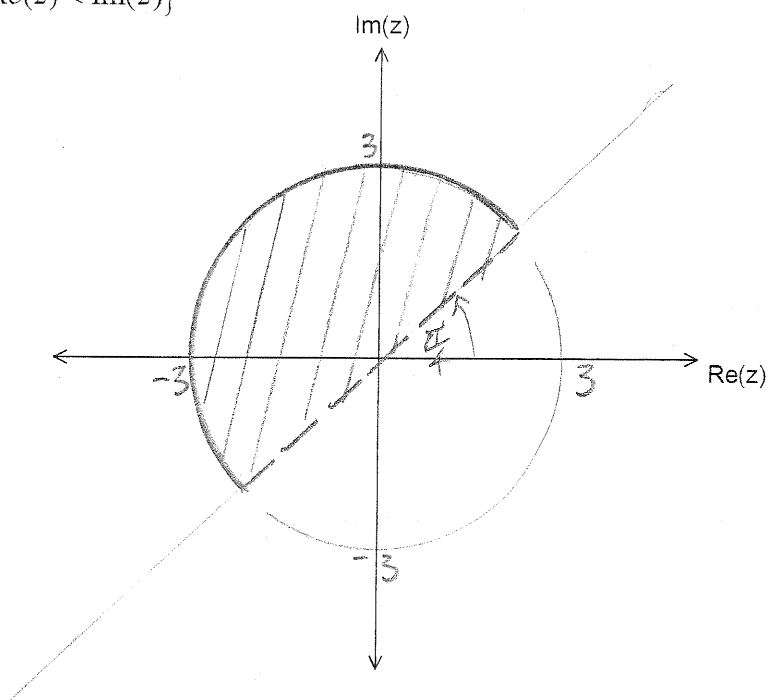
$z - \bar{z} = 2iy$

$\therefore y = 1$



b) $\{z : |z| \le 3\}$ and $\{Re(z) < \text{Im}(z)\}$

circle
line } ✓
shading ✓
boundary ✓



11. [3 marks]

Points M and N have position vectors $\begin{pmatrix}-1\\2\\5\end{pmatrix}$ and $\begin{pmatrix}13\\23\\-2\end{pmatrix}$ respectively. Find the position vector of the point that divides MN internally in the ratio 2:5.

Let P be the point:

$$\vec{OP} = \vec{OM} + \frac{2}{7}\vec{MN}$$
$$= \begin{pmatrix}-1\\2\\5\end{pmatrix} + \frac{2}{7}\begin{pmatrix}14\\21\\-7\end{pmatrix}$$
$$= \begin{pmatrix}3\\8\\3\end{pmatrix}$$

12. [6 marks]
Determine the shortest distance between the point P with position vector $2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and the plane $r.(\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) = 17$.

Let Q with pv $\begin{pmatrix}a\\b\\c\end{pmatrix}$ be the nearest point to P on the plane.

Then $\vec{QP} = \begin{pmatrix}2\\1\\-2\end{pmatrix} - \begin{pmatrix}a\\b\\c\end{pmatrix} = k\begin{pmatrix}1\\-1\\3\end{pmatrix}$ (since $\perp$ to plane.)

$$\therefore \begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}2-k\\1+k\\-2-3k\end{pmatrix}$$

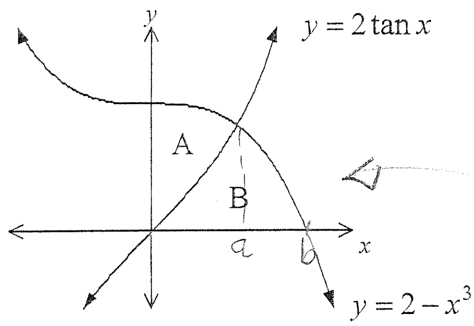
Now $\begin{pmatrix}2-k\\1+k\\-2-3k\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\3\end{pmatrix} = 17$ (As Q is on the plane)

$$1(2-k) - 1(1+k) + 3(-2-3k) = 17$$
$$k = -2$$

$$\therefore \vec{QP} = -2\begin{pmatrix}1\\-1\\3\end{pmatrix}$$

$$\therefore \text{Min dist} = |\vec{QP}|$$
$$= 4\sqrt{11}$$

13. [5,2 marks]



Let A and B be the regions in the first quadrant shown in the figure above. The region A is bounded by the y-axis and the graphs $y = 2\tan x$ and $y = 2 - x^3$. The region B is bounded by the x-axis and the graphs of $y = 2\tan x$ and $y = 2 - x^3$.

(TO 3DP)

a) Find the area of region B. (SEE DIAGRAM)

$$A_B = \int_0^a (2\tan x)\,dx + \int_a^b (2 - x^3)\,dx$$ ✓✓

where $a = 0{\cdot}6943266$
$b = \sqrt[3]{2} = 1{\cdot}25992105$ ✓

$A_B = 1{\cdot}086$ (3dp) UNITS^2 ✓✓

(ANSWER ONLY MAX 2/5)

(TO 3DP)

b) Find the area of region A.

$$A_A = A_{A+B} - A_B$$

$$= \int_0^b (2 - x^3)\,dx - A_B$$

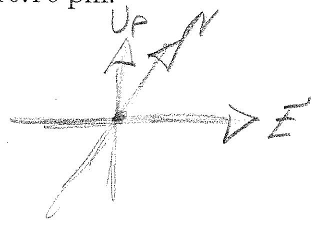
$= 0{\cdot}804$ (3dp) UNITS^2 ✓✓ (ANSWER ONLY OK)

14. [1,2,3,4 marks]

A fighter jet is maintaining a constant velocity of $\left(\underset{\sim}{a} = 60\underset{\sim}{i} + 80\underset{\sim}{j} + 4\underset{\sim}{k}\right)$ m/s, with the unit vectors representing directions East, North and Upwards respectively. Sea level is zero altitude. The fighter jet is being tracked from a mountain base 1250m above sea level. At 10 pm the jet is 6km East and 4km South of the base and its altitude is 2250m above sea level.

35 West 60

a) Find the position vector of the jet with respect to the base at 10.10 pm.



$$\underset{\sim}{r} = \begin{pmatrix} -35000 \\ -60000 \\ 1000 \end{pmatrix} + 600\begin{pmatrix} 60 \\ 80 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -12 \\ 3.4 \end{pmatrix} \text{ km}$$

b) At what time is the jet due East of the base?

Due East when j comp is zero

$$0 = -60000 + 80t$$

$$t = 750 \text{ s}$$

DUE EAST AT 10:12:30 pm

c) How far is the jet from the base at 10.15pm?

$$|\underset{\sim}{r}| = \left\| \begin{pmatrix} 19 \\ 12 \\ 4.6 \end{pmatrix} \right\| = \sqrt{19^2 + 12^2 + 4.6^2}$$

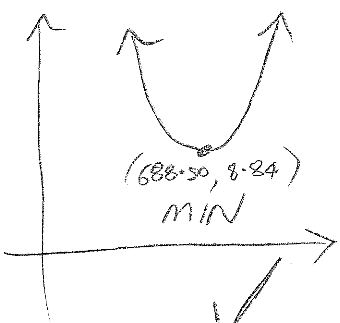
$$= \frac{\sqrt{13154}}{5}$$

$$= 22.9 \text{ km (1dp)}$$

d) Find the least distance between the jet and the base.

Let least distance be L.

$$L^2 = (60t - 35000)^2 + (80t - 60000)^2 + (1000 + 4t)^2$$



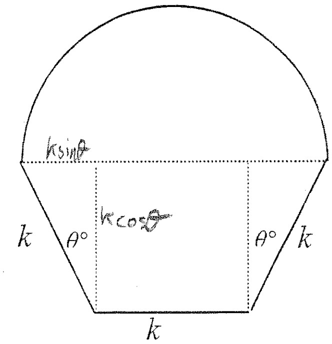
$$\frac{dL^2}{dt} = 120(60t - 35000) + 160(80t - 60000) + 8(1000 + 4t)$$

$$0 = 20032t - 13792000$$

$$\therefore t = 688.50 \text{ s (2dp)}$$

$$L = \left\| \begin{pmatrix} 6.31 \\ -4.92 \\ 3.75 \end{pmatrix} \right\| = 8.84 \text{ km (2dp)}$$

15. [4,6,2 *marks*]
The shape shown below consists of a trapezium with a semi-circle on top. The three straight edges are of fixed length k metres and the angle the two straight sides make with the vertical is $\theta°$ where $0° \le \theta \le 90°$.



a) Show that the area, A, of the shape is given by:

$$A = \frac{k^2}{8}\left\{\pi(1+2\sin\theta)^2 + 4\sin(2\theta) + 8\cos\theta\right\}$$

$$\text{Area Trap} = \frac{k\cos\theta}{2}\left(2k\sin\theta + k + k\right)$$
$$= k\cos\theta\left(k\sin\theta + k\right)$$
$$= k^2\cos\theta\sin\theta + k^2\cos\theta$$

$$\text{Area Semi Circle} = \frac{\pi}{2}\left(\frac{2k\sin\theta + k}{2}\right)^2$$
$$= \frac{\pi}{2}\left(\frac{k(2\sin\theta+1)}{2}\right)^2$$
$$= \frac{\pi k^2}{8}(2\sin\theta+1)^2$$

$$A = \frac{\pi k^2}{8}(2\sin\theta+1)^2 + k^2\cos\theta\sin\theta + k^2\cos\theta$$
$$= \frac{k^2}{8}\left\{\pi(1+2\sin\theta)^2 + 8\cos\theta\sin\theta + 8\cos\theta\right\}$$
$$= \frac{k^2}{8}\left\{\pi(1+2\sin\theta)^2 + 4\sin 2\theta + 8\cos\theta\right\}$$

b) Using calculus, show that the area is optimised if $\dfrac{\sin\theta - \cos(2\theta)}{\cos\theta + \sin(2\theta)} = \dfrac{\pi}{2}$

$$\frac{dA}{d\theta} = \frac{K^2}{8}\left\{\pi(4\cos\theta)(1+2\sin\theta) - 8\cos 2\theta - 8\sin\theta\right\}$$

Optimum Area: $0 = 4\pi\cos\theta(1+2\sin\theta) + 8(\cos 2\theta - \sin\theta)$

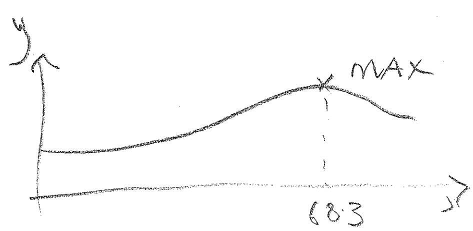
$$-8(\cos 2\theta - \sin\theta) = 4\pi(\cos\theta + 2\cos\theta\sin\theta)$$

$$\frac{8(\sin\theta - \cos 2\theta)}{4(\cos\theta + \sin 2\theta)} = \pi$$

$$\frac{\sin\theta - \cos 2\theta}{\cos\theta + \sin 2\theta} = \frac{\pi}{2}$$

c) Determine the value of θ that maximises the area.

$\theta = 68.3°$ (1 dp)



NOTE: $A_{MAX} \approx 3.92K^2$
This is not required